Written exam statistic

Task 1: Equipp the set $\Omega = \mathbb{N}$ with a probability measure $\mathbb{P}$. Here $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, ...\}$.

a) define a random variable X on the probability space $(\Omega, \mathbb{P})$ such that $0 < var(X) < \infty$

b) compute the expectation and variance of X

c) Find two independent events A and B on $(\Omega, \mathbb{P})$

Task 2: Given a continuos random variable X with cdf (cumulative distribution function) $F_X(z) = 1 - e^{-ax^2}$ for $x \geq 0$ and zero otherwise ($a > 0$)

a) Compute the density $f(z)$ of X and draw $f(z)$ and $F(z)$ for the value $a = 1$

b) Compute for $a = 1$ the probabilities $\Pr\{X \leq -1\}$, $\Pr\{X = 1\}$, $\Pr\{X \geq 0\}$ and $\Pr\{1 < X\}$

Task 3: Let X be a random variable with density $f(z) = \frac{1}{\sqrt{z}}$ for $0 < z \leq 1$ and zero otherwise.

a) Compute the cdf $F(z)$ of X and draw f and F

b) Compute the expectation of X

Task 4: Given a Poisson point process with intensity $\lambda = \frac{1}{2}$. What is the probability that a realization of the process has no points in the interval $[3, 5] \cup [-2, 0]$?

Task 5: Let $X_1, ..., X_n$ be iid random variables with values in $\{-2; +2\}$ and $\Pr\{X_i = 2\} = \frac{1}{2}$. Let $S_n = \sum_{i=1}^{n} X_i$.

a) Use the central limit theorem to estimate the probability that $|S_{40000}| > 800$.

($\Phi(1) = 0.841\,34; \Phi(1.5) = 0.933\,19; \Phi(2) = 0.977\,25;\ \Phi(2.5) = 0.993\,79; \Phi(3) = 0.998\,65; \Phi(4) = 0.999\,97$)

b) (optional): Use the Rademacher bound to obtain an upper bound on the probability $\Pr(|S_{40000}| > 800)$. The Rademacher bound is as follows: Let Y_i be iid random variables with values $Y_i \in \{-1, +1\}$ and $\Pr\{Y_i = 1\} = \frac{1}{2}$. Let $\{a_i\}_{i=1}^{n}$ be any sequence of real numbers . Then the following bound holds:

$$\Pr\left\{\sum_{i=1}^{n} a_i Y_i \geq t \cdot \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\right\} \leq e^{-\frac{t^2}{2}} \tag{1}$$

.

Task 6: Given $X_1, ..., X_n$ iid random variables with $X_i \sim U([0, \theta])$ with unknown θ and U being the uniform distribution.

Use the method of moments to estimate θ if you have a sample of 5 values $X_1 = 1$, $X_2 = 3$, $X_3 = 2$, $X_4 = 1.5$ and $X_5 = 2.5$.

Task 7: The joint density f of three random variables X, Y, Z is given as $f(x, y, z) = 4a^5 xye^{-(2ax+ay+az)}$ for $x, y, z \geq 0$ and zero otherwise; the parameter $a > 0$ is unknown. For the sample $(X, Y, Z) = (1, 2, 2)$ compute the maximum likelihood estimator of a.

Task 8: Given a random variable X with uniform distribution on $[0, 2]$, that is $X \sim U([0, 2])$. Compute the cdf and the density of $Y = X^2$.

Task 9: : Given two independent random variables X and Y with expectations $EX = 1$ and $EY = 2$ and variances $VarX = 2$ and $VarY = 4$. Use the Chebychev inequality to estimate the probability $\Pr\{|2X - Y| \geq 4\}$ from above.

Task 11: (optional) Let X be a uniform distributed continuos random variable in the interval $[0; 1]$:

Compute the density and cumulative distribution function of the random variable $Y = \ln X$: Sketch the graph of the density and the cumulative distribution function of Y: